Optimum Criterion for Multilevel Coding Systems in AWGN Channels*

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Abstract-Based on [1,2,3], a novel criterion which is "capacity rule" and "mapping strategy " for the design of optimal MLC scheme over AWGN channels is proposed. According to this theory, the performance of multilevel coding with multistage decoding schemes (MLC/MSD) in AWGN channels is investigated, in which BCH codes are chosen as component codes, and three mapping strategies with 8ASK modulation are used. Numerical results indicate that when code rates of component codes in MLC scheme are designed based on "capacity rule", the performance of the system with UP is optimum for AWGN fading channels.

I. Introduction

Coded modulation methods over the AWGN channel have been studied extensively for years starting with early work by Ungerboeck (trellis coded modulation (TCM)) [4],[5] and by Imai/Hirakawa (multilevel coding (MLC)) [6]. The common core is to optimize the code in Euclidean space rather than dealing with Hamming distance as in classical coding schemes.

The code parameters of TCM approach are chosen by means of an exhaustive computer search in order to maximize the minimum distance of the coded sequences in Euclidean space. And the selection of code rate of TCM isn't flexible. In contrast to TCM, the MLC approach provided flexible transmission rates, because it decouples the dimensionality of the signal constellation from the code rate. But in practice, system performance was severely degraded due to high error rates at low levels. A lot of effort was devoted to overcome this effect.

For practical coded modulation schemes where boundary effects have to be taken account, Huber et al [9] independently proved that the capacity of the modulation scheme can be achieved by multilevel codes together with multistage decoding if and only if the individual rates of the component cods are properly chosen. In this paper, based on the calculation of capacities of equivalent channels for MLC/MSD scheme over AWGN channel, a novel criterion for optimal MLC scheme suitable to AWGN channels is proposed by investigating the performance of MLC/MSD scheme with 8ASK modulation, in which different mapping strategies and BCH codes as component codes are used.

II. MLC Scheme and Capacity Rule

- **Multilevel coding scheme**

MLC is based on the mapping by set partitioning. Therefore, via a binary set partitioning of the signal set $\mathbf{A}=\{a_m \mid m\in \{0, 1, 2,\ldots,2^{l-1}\}\}$, a mapping $\mathbf{m}\leftrightarrow\mathbf{C}$ of binary labels $\mathbf{C}=(c^0, c^1, \ldots, c^{l-1})$, $c^i\in \{0, 1\}$, $i\in \{0, 1,\ldots l-1\}$, to signal points a_m is defined. The subsets of signal points at level i are denoted by the path in the set partitioning tree from the root to the subsets, i.e.

$$\mathbf{A}_{c^0\ldots c^i} = \{a_m | m \leftrightarrow (c^0, c^1, \ldots c^i, x^{i+1}, \ldots x^{l-1}),\ x^j\in \{0, 1\},\ j\in \{i+1,\ldots,l-1\}\} \quad (1)$$

For conciseness, we restrict our considerations to MLC schemes with binary component codes. A sequence $\langle q\rangle_1^k$ of source data symbols is demultiplexed in l sequences $\langle q^i\rangle_1^{K_i}$ ($i \in \{0,1,\ldots l-1\}$, $\sum_i K_i = K$), which are fed into individual binary encoders. Then, the encoders produce code sequences $\langle c^i\rangle_1^n$ of uniform length n. The resulting binary labels $c_\mu=(c_\mu^0, \ldots c_\mu^{l-1})$, μ=1, ... n, are mapped to signal points a_{m_μ}

Therefore, the code rate R of the MLC scheme is:

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$$R = \frac{k}{n} = \sum_{0}^{l-1} R_i = \frac{\sum_{i=0}^{l-1} k_i}{n} \quad (2)$$

For component codes c^i at level i, block codes, concatenated codes or even no code can be chosen. In this paper we select BCH codes as component codes and take three levels for our discussion, i. e. l=3.

- **Capacity rule**

Since the mapping M is bijective and hence lossless in the sense of information theory, the mutual information $I(Y;A)$ between the transmitted signal point $a \in A$ and the received signal $y \in Y$ equals the mutual information $I(Y;X^0,X^1,\ldots X^{l-1})$ between the address vector $X \in \{0,1\}^l$ and the received signal point:

$$I(Y;A) = I(Y;X^0,X^1,\ldots X^{l-1}) \quad (3)$$

Where, we denoted the random variables corresponding to the transmitted and received signal point, the binary address vector, and its components by capital letters.

Then, we applying the chain rule to the mutual information yields[10]

$$I(Y;X^0,X^1,\ldots X^{l-1}) = I(Y;X^0) + I(Y;X^1|X^0) + \ldots + I(Y;X^{l-1}|X^0X^1\ldots X^{l-2}) \quad (4)$$

This equation may be interpreted in the following way: the transmission of vectors with binary digits $X^i, i = 0,1,\ldots, l-1$, over the physical channel can be virtually separated into the parallel transmission of the digits x^i over ℓ equivalent channels. The equivalent channel i consists of the equivalent mapper i, provided that the digits $x^0\ldots x^{i-1}$ and the noise channel are known. The binary symbol x^i is multiply represented in the signal set of the equivalent mapper i for $i < \ell - 2$.

From the chain rule the mutual information of the equivalent channel i can be calculated by (5):

$$I(Y;X^i\ldots X^{l-1}/X^0\ldots X^{i-1}) = I(Y;X^i/X^0\ldots X^{i-1}) + I(Y;X^{i+1}\ldots X^{l-1}/X^0\ldots X^i) \quad (5)$$

However, the capacity C^i for given a-priori probabilities of signal points yields is :

$$C^i = I(Y;X^i/X^0\ldots X^{i-1}) = I(Y;X^i\ldots X^{l-1}/X^0\ldots X^{i-1}) - I(Y;X^{i+1}\ldots X^{l-1}/X^0\ldots X^i) \quad (6)$$

where the mutual information $I(Y;X^i\ldots X^{l-1}/X^0\ldots X^{i-1})$ is calculated by averaging with respect to all possible combinations of $x^0,\ldots, x^{i-1}$:

$$I(Y;X^i\ldots X^{l-1}/X^0\ldots X^{i-1}) = E_{x^0\ldots x^{i-1} \in \{0,1\}^i}\{I(Y;X^i\ldots X^{l-1}/X^0\ldots X^{i-1})\} \quad (7)$$

Thus the capacities of the equivalent channel i can be got[9]:

$$\begin{cases} C^i = E_{x^0\ldots x^{i-1}}\{C(A(x^0\ldots x^{i-1}))\} - E_{x^0\ldots x^i}\{C(A(x^0\ldots x^i))\}, & i = 1,\ldots,l-1 \\ C^0 = C(A) - E_{x^0}\{C(A(x^0))\} & i = 0 \end{cases} \quad (8)$$

Based on the concept of the equivalent channels and its capacities, we can easily draw our "capacity rule" or "rate rule". Given a 2 -ary digital modulation scheme, choose the rate R^i at the individual coding level i of a MLC scheme to equal the capacity C^i of the equivalent channel i :

$$R^i = C^i \qquad i = 0,1,\ldots, l-1 \quad (9)$$

The basis of the capacity rule is to characterize the transmission properties of the equivalent channels by its capacities. Operating at the capacity limit of MLC scheme, the capacity rule provides the maximum individual rates to be transmitted with arbitrarily low error probability. Thus, the design of MLC scheme with an optimum trade-off between power and bandwidth efficiency has to be based on the capacity rule.

The capacity of any signal (sub-) set B with $|B|$ equiprobable elements is: [9]

$$C(B) = \int_Y \sum_{a_m \in B} f_{y|a_m}(y) \log_2\left(\frac{f_{y|a_m}(y)}{\frac{1}{|B|}\sum_{a_m \in B} f_{y|a_m}(y)}\right)dy \quad (10)$$

Where $f_{y|a_m}(y)$ is the probability density function of the channel considered. Note that we can get different capacity results when we submit different channel characteristics into above equation (10).

III. Different rates of Three Mapping Strategies

Traditional Ungerboeck partitioning (UP) proposed by [5] is aimed at maximizing the intra subset minimum Euclidean distance. As an inverse strategy, we call block partitioning (BP), which maintains the intra subset minimum Euclidean distance. Last strategy is called mixed partitioning (MP) which is a kind of combination of UP and BP strategies. Taking 8ASK signal constellation for example, BP rule[9] is shown in Fig.1.

MP results from a combination. In this letter, it is defined in this way: BP-UP-UP which means that the first partitioning step is done by the rule of BP and followed by UP and UP.

From the calculation results of capacities for three level MLC schemes with different set partitioning strategies[1], the rate design values of MLC/MSD schemes with 8ASK modulation are listed in Table I for AWGN channel. The total code rate is 2.5bits/symbol.

IV. Results and Discussions

According to results in Table I, the performance of MLC/MSD scheme in 8ASK signal constellation over AWGN channels is investigated, in which BCH codes with code lengths of 127 are used as component codes. Simulations results are shown in Fig2.~Fig.5. There are three MLC/MSD schemes in Fig.2~Fig.4, i.e., CODE1, CODE2 and CODE3. For CODE2, the rates of binary component codes on three levels are designed according to "capacity rule" shown in Table I, while "capacity rule" is not obeyed for CODE1 and CODE3. The total rates of three schemes are all chosen as: R=2.5bits/symbol. In Fig.5 the performance of MLC schemes with three set partitioning strategies according to individual "rate rule" is compared. From the simulation results, we can see:

(a)The performance of MLC/MSD scheme according to "capacity rule" is optimal at same bandwidth efficiency. As Fig.2, Fig.3 and Fig.4 show, CODE2 scheme is superior to CODE1 and CODE3 schemes.

(b)Schemes, which isn't obey "rate rule", have different affection to performance at the same total rates. As CODE1 scheme in Fig.2 shows, the rate R_0 of the first level is greater than the capacity, R_1 of the second level is lower than the capacity, therefore, the performance of CODE1 is inferior to CODE2 scheme. In CODE1 scheme of Fig.3, the rate R_0 equals to the capacity, R_1 is lower and R_2 is greater than capacity, therefore, the performance of CODE1 scheme is nearly the same as that of CODE2 scheme. Thus, the rate of the first level R_0 must be designed according to the capacity of equivalent channel comparing with other levels. If R_0 is greater than the capacity, the performance will degrade more greatly.

(c)Fig.5 shows that the performance of MLC system with UP is optimal compared with BP and MP according to individual "capacity rule". Therefore, a novel criterion of MLC scheme suitable to AWGN channels, which is "capacity rule" and "UP", is proposed in this paper. Fortunately, this criterion is completely identical to Euclidean metric proposed by Ungerboeck. That is to say, when code rates of component codes in MLC scheme are designed based on capacity rule, the system with UP has optimum performance for AWGN channels.

V. Conclusions

A kind of "novel criterion", which is "capacity rule" and "UP strategy", for the design of optimum MLC system applied to AWGN channels is proposed in this paper by investigating the performance of MLC/MSD system. That is to say, when code rates of component codes in MLC system are designed based on "capacity rule", and UP strategy is used in AWGN channels, the system performance will be optimal. Fortunately, this criterion is identical to the Euclidean metric of good codes for CM schemes including MLC and TCM.

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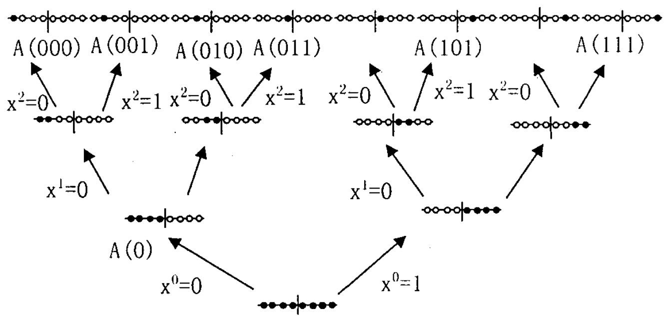
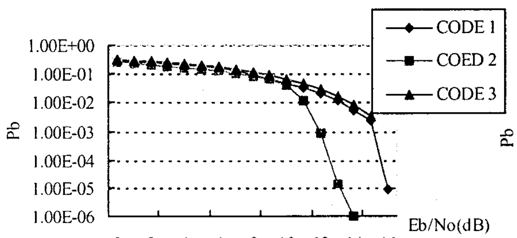
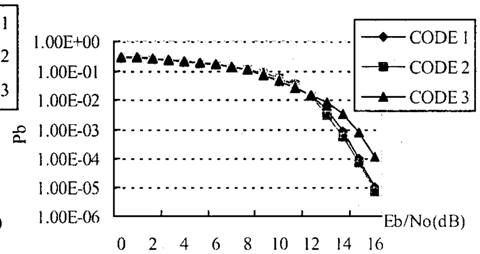


Table I
Different rates of MLC/MSD with 8ASK modulation based on capacity rule (R=2.5bits/symbol)

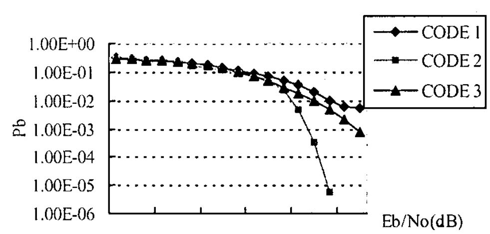
	UP	BP	MP
AWGN Channel	$C_0=R_0=0.5$ $C_1=R_1=1$ $C_2=R_2=1$	$C_0=R_0=0.95$ $C_1=R_1=0.85$ $C_2=R_2=0.7$	$C_0=R_0=0.875$ $C_1=R_1=0.625$ $C_2=R_2=1$



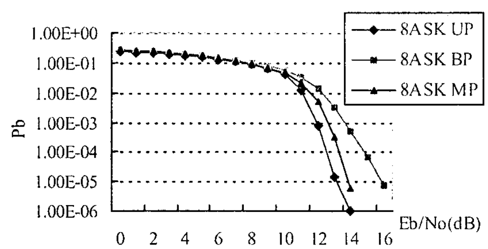
CODE1-C0:(127,127);C1:(127,57,11);C2:(127,127)
CODE2-C0:(127,57,11);C1:(127,127);C2:(127,127)
CODE3-C0:(127,127);C1:(127,127);C2:(127,57,11)
Fig.2 Performance comparison of different rates (AWGN channel, UP, MLC/MSD)



CODE1-C0:(127,120,1);C1:(127,85,6);C2:(127,106,3)
CODE2--C0:(127,120,1);C1:(127,106,3);C2:(127,85,6)
CODE3--C0:(127,106,3);C1:(127,85,6);C2:(127,120,1)
Fig.3 Performance comparison of different rates (AWGN channel, BP, MLC/MSD)



CODE1-C0:(127,78,7);C1:(127,106,3);C2:(127,127)
CODE2--C0:(127,106,3);C1:(127,78,7);C2:(127,127)
CODE3--C0:(127,106,3);C1:(127,127);C2:(127,78,7)
Fig.4 Performance comparison of different rates (AWGN channel, MP, MLC/MSD)



UP--C0:(127,57,11);C1:(127,127);C2:(127,127)
BP--C0:(127,120,1);C1:(127,106,3);C2:(127,85,6)
MP--C0:(127,106,3);C1:(127,78,7);C2:(127,127)
Fig.5 Performance comparison of different mapping strategies (AWGN channel, MLC/MSD)